A Model of an N-Body System at Fixed Electrical Potentials

Jared Kirschner, Noah Tye

***Abstract*—We discuss the creation of a model which can be used to dynamically simulate a system of n-bodies at fixed electrical potentials within any defined boundary. We will also describe how this model can be discretized and solved numerically.**

I. Introduction

MANY questions in electromagnetism require one to solve for the electric potential and/or electric field in a defined segment of space. Some solutions, such as integral formulations, are only useful for simplistic situations. The resulting integral often cannot be evaluated analytically due to its complexity and will only work if the rest of the universe is empty (in other words, the boundary conditions are assumed to be zero at infinity). In cases of high symmetry, we can solve for the electric field with Gauss' Law much more quickly than with an integral formulation. However, Gauss' Law also assumes that the rest of the universe is empty, or can only be accurate over small regions of space. By using a complete basis set and Fourier's "trick," one can derive an equation describing the voltage between defined boundaries (from which the electric field can be derived). However, such an approach is only feasible in cases with simple boundaries. Herein we shall develop a model which can be used to model and behavior of any boundary condition containing any number of bodies. This approach will use a discrete model which can be described by a matrix equation. This matrix equation can this be solved numerically to show the voltage and electric field at any point.

II. Governing Equations

According to Poisson's Equation, the net curvature of the potential at an arbitrary region is $\nabla^2 V = \frac{\rho}{\epsilon_0}$, where V is the electrical potential, ρ is the charge density, and ϵ_0 is the permittivity of free space. In regions without charge, we have Laplace's Equation:

$$\nabla^2 V = 0 \tag{1}$$

We can then derive the electric field from the potential difference through the following definition:

$$E = -\nabla V \tag{2}$$

J. Kirschner and N. Tye are undergraduate students at Franklin W. Olin College of Engineering, 1000 Olin Way, Needham, MA 02492. E-mail (respectively): Jared.Kirschner@Students.olin.edu and Noah.Tye@Students.olin.edu.

Lastly, we must consider the force acting on each object from its interactions with other objects and the boundaries. Each object of fixed potential will tend to move towards the region with the strongest electric field. The electrostatic pressure at each point is equal to:

$$\vec{P} = \int \vec{F} dA = \int \frac{1}{2}\epsilon_0 |\ E\ |^2 dA\ \ \hat{n} \tag{3}$$

...where $\vec{P}$ is the pressure, $\vec{F}$ is the force, dA is the differential area, ϵ_0 is the permittivity of free space, and $\hat{n}$ is the normal unit vector to the surface. By integrating the contributions to the electrostatic pressure along the entire surface of the object and then dividing by the surface area, we obtain the net force. We can then divide this force vector by the mass of each object in order to obtain its acceleration vector.

III. The Model

In order to solve Laplace's equation (equation 1), we will formulate a matrix equation which can be solved in matrix. For a position $p_{n,m}$, the solution to Laplace's equation can be expressed as:

$$\nabla^2 V_{n,m} = \frac{V_{n-1,m} + V_{n+1,m} + V_{n,m-1} + V_{n,m+1}}{4} \tag{4}$$

In order to determine the general formula for the matrix equation, we can write out the equations for all points within a small grid of $n \cdot m$ internal points and an outer boundary (top $t_{1:n}$, left $l_{1:m}$, bottom $b_{1:n}$, right $r_{1:m}$). For example, point 1,1 will have the following equation: $\frac{V_{t_1}+V_{l_1}+V_{2,1}+V_{2,1}}{4} = 0$. All constants will move to the right side of the equation; in the given example equation, the two boundary conditions will both be constants and move to the right side. If either point $V_{1,2}$ or $V_{2,1}$ is defined, these will also move to the right side of the equation. We then place the elements on the left side of the equation into an $n \times m$ diagonal matrix, D, and the right side of the equations into a columnwise vector of $n \cdot m$ elements, b. By dividing D into b, we solve for a columnwise vector v of potentials where the indicies $1{:}n \cdot m$ correspond to indicies associated with each subscripted point.

After writing out a small $(x \times y)$ matrix by hand, one can derive the following general patterns:

1) The central diagonal is always equal to -1.
2) The ± 1 diagonals have a pattern such that there is a repeated unit where the first $x-1$ elements are $\frac{1}{4}$, and the $(x-1)^{th}$ element is 0.

3) The $\pm x$ diagonals are always equal to $\frac{1}{4}$.

We can also write out the general pattern for the columnwise vector b using the same assumptions:

1) Top Row: For the first $lenX^{th}$ elements, we will add a constant of $-\frac{V_{t_n}}{4}$ where n is the current position in the array of appropriate elements.
2) Bottom Row: For the last row $(x \cdot (y-1)+1 : x \cdot y$ elements), we will add a constant of $-\frac{V_{b_n}}{4}$ where n is the current position in the array of appropriate elements.
3) Left Row: For every index $i \cdot x + 1$ where i is any non-negative integer below y, we will add a constant of $-\frac{V_{l_{i+1}}}{4}$.
4) Right Row: For every index $i \cdot x$ where i is any positive integer value less than or equal to y, we will add a constant of $-\frac{V_{r_i}}{4}$.

Lastly, to account for positions within the grid that one wishes to define, change all points which had been defined as variables in the matrix D to zero, and move the defined values into the columnwise vector b as dictated by the equations. For example, if a point is defined within a row of the matrix to have a coefficient of $\frac{1}{4}$, this coefficient will have to be applied to the constant (and the sign changed, reflecting moving the constant to the other side of the equation) to move it accurately into the columnwise vector b of constants. As this element has now been defined, the row corresponding to its index will be removed from the matrix and the columnwise vector.

After performing this operation on all defined points, the matrix equation can be solved numerically in one step. However, as the matrix has dimensions of $(n \cdot m)^2$, one will find that such a computation can be fairly memory intensive. In order to vastly reduce the memory usage of the program, one can use sparse matrices, which essentially map subscripts to all non-zero elements of a matrix. As the matrix we have created is mostly zeros, a very high compression (between 10^2 and 10^3) can be achieved.

The solution to this matrix equation will yield the potential at all internal points of the grid. Using equation 2, we can solve for the electric field at all internal points. With the electric field, we can calculate the net force on the surface of an object of defined potential using equation 3. Through these methods, one can not only calculate the field for n bodies, but simulate their motion through time.